

Problem 1.15

Calculate the divergence of the following vector functions:

(a) $\mathbf{v}_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$.

(b) $\mathbf{v}_b = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$.

(c) $\mathbf{v}_c = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$.

Solution

Take the divergence of each of the given functions.

Part (a)

$$\begin{aligned}\nabla \cdot \mathbf{v}_a &= \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) \cdot (x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}) \\ &= (\hat{\mathbf{x}} \cdot \hat{\mathbf{x}}) \frac{\partial}{\partial x}(x^2) + (\hat{\mathbf{y}} \cdot \hat{\mathbf{y}}) \frac{\partial}{\partial y}(3xz^2) - (\hat{\mathbf{z}} \cdot \hat{\mathbf{z}}) \frac{\partial}{\partial z}(2xz) \\ &= (1)(2x) + (1)(0) - (1)(2x) \\ &= 0\end{aligned}$$

Part (b)

$$\begin{aligned}\nabla \cdot \mathbf{v}_b &= \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) \cdot (xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}) \\ &= (\hat{\mathbf{x}} \cdot \hat{\mathbf{x}}) \frac{\partial}{\partial x}(xy) + (\hat{\mathbf{y}} \cdot \hat{\mathbf{y}}) \frac{\partial}{\partial y}(2yz) + (\hat{\mathbf{z}} \cdot \hat{\mathbf{z}}) \frac{\partial}{\partial z}(3zx) \\ &= (1)(y) + (1)(2z) + (1)(3x) \\ &= y + 2z + 3x\end{aligned}$$

Part (c)

$$\begin{aligned}\nabla \cdot \mathbf{v}_c &= \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) \cdot [y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}] \\ &= (\hat{\mathbf{x}} \cdot \hat{\mathbf{x}}) \frac{\partial}{\partial x}(y^2) + (\hat{\mathbf{y}} \cdot \hat{\mathbf{y}}) \frac{\partial}{\partial y}(2xy + z^2) + (\hat{\mathbf{z}} \cdot \hat{\mathbf{z}}) \frac{\partial}{\partial z}(2yz) \\ &= (1)(0) + (1)(2x) + (1)(2y) \\ &= 2x + 2y\end{aligned}$$